GAUGE–FIXING INDEPENDENCE OF TEST FIELDS IN YANG–MILLS THEORIES

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Abstract

We derive a generalized Nielsen identity for the case of Yang-Mills theories that include some classical fields. We discuss under which circumstances the effective action of the classical fields (i.e., after integration of quantum fields) becomes gauge–fixing independent. We conclude that classical test fields provide a physical insight into the problem of the gauge–fixing dependence of the quantum effective action.

I. INTRODUCTION

As it is well known, whatever the scheme employed to quantize a gauge theory is (i.e., Gupta-Bleuler, Fadeev–Popov, BRST, Batalin–Vilkovisky methods), it is necessary to fix a particular gauge in order to carry on the quantization program. Gauge–fixing is implemented introducing into the classical action an additional non–invariant term, the so called gauge–breaking term. The resulting classical effective action is a gauge–parameter dependent functional on the field–configuration manifold. Required gauge invariance of the quantum theory follows, in turn, from the fact that expectation values of physical magnitudes become independent of the choice of this particular term.

The best way to understand how this gauge independence appears is by noticing that when estimating S-matrix elements or expectation values of gauge independent magnitudes one makes use of the quantum effective action on–shell, i.e., evaluated at those configurations that extremize it. According to Nielsen identities [1], the variation of the quantum effective action due to changes in the functions that fix the gauge is linear in the quantum–corrected equations of motion for the mean fields. It immediately follows that the on–shell quantum efective action does not depend on the choice of the gauge–breaking term. The mean fields do depend on the gauge–fixing, but this dependence exactly cancels out the explicit gauge–fixing dependence of the quantum effective action [2].

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In some situations one is interested in the dynamical evolution of the fields rather than in the S-matrix elements. We mention some examples: the analysis of phase transitions in field theory and condensed matter physics [3], the non-equilibrium aspects of the quark-gluon plasma [4], the quantum corrections to solitons [5], and the quantum corrections to the geometry in semiclassical gravity, relevant in the early universe and black hole physics [6]. In all these cases the gauge–fixing dependence of the mean fields becomes a problem. This dependence does not contradict the gauge–fixing independence of mean values of gauge invariant operators, since these are in general non linear functions of the fields, and it is not possible to compute them directly from the mean values of the fields.

One way to by–pass this difficulty is through a redefinition of the quantum effective action. Such is, for instance, the case of the Vilkovisky–De Witt formalism [7]. Vilkovisky and De Witt introduced a connection on the field–manifold and they used it in order to obtain a modified gauge–independent expression for the quantum effective action. However, despite of the gauge independence of the Vilkovisky–De Witt effective action, it depends on the choice of the connection on the configuration space, and there is not a universal criteria for the determination of such a connection, particularly in the case of interacting fields [8]. In that sense, the Vilkovisky–De Witt definition is not a real solution for the problem of the gauge–fixing dependence of the mean fields.

Another possibility lies of course in considering effective potentials or even actions for gauge–invariant operators [9]. In this framework, the issue of renormalization is not completely clear. It is also possible to construct an effective potential which is a gauge–invariant function of a gauge–invariant order parameter. This approach has been developed in equilibrium [10] and also in non–equilibrium situations [11], for the abelian Higgs model. Extension to non–abelian theories seems not straightforward.

In this paper we present an alternative way to approach the subject, following the ideas exposed in [12], where the gauge–fixing problem was analyzed in the context of semiclassical gravity. In that context, due to graviton corrections, the background metric that solves the semiclassical Einstein equations is gauge–fixing dependent. However, the metric can be "measured" by analyzing the trajectory of a classical test particle, which plays the role of a classical device. The coupling of the particle to gravitons compensates the gauge–fixing dependence of the metric, and the trajectory of the particle becomes gauge–fixing independent.

In the present letter we will find similar results for a Yang-Mills theory. Instead of introducing a new definition for the quantum effective action, we will consider a field theory involving a set of classical test fields interacting with a quantum background. We study particularly the case of classical matter test fields interacting with a pure Yang–Mills environment. In that context, test fields could be interpreted as a classical device "measuring" physical quantities depending on the Yang–Mills degrees of freedom.

BRST–invariance of the classical effective action, when combined with the constraints imposed by the classical nature of the device, yields a set of "generalized" Nielsen identities expressing how does the dynamical evolution of the test fields depends on the choice of the gauge–fixing conditions. As it will be shown, the evolution of the classical device does not depend on the gauge–breaking term, even when the equations of motion of the quantum degrees of freedom do so. This problem has also been analyzed by Kazakov and Pronin [13] using a different approach, and we will comment on this in the conclusions.

The paper is organized as follows. We first review how the Yang–Mills gauge system could be quantized according to the BRST–quantization procedure. Then, we derive the generalized Nielsen identity which governs, as we have mention, the gauge dependence of the quantum effective action. Finally, we explicitly show how the dynamics of the test fields is gauge–fixing independent.

II. A GENERALIZED NIELSEN IDENTITY

We will consider a Yang-Mills theory coupled to classical matter fields. Since we are interested in the dynamical evolution of the classical fields (considered as a classical device degrees of freedom), it will be useful to decompose the action into two terms[1]:

$$\mathcal{S}\left[\phi, A\right] = \mathcal{S}_{\mathrm{CD}}\left[\phi\right] + \mathcal{S}_{\mathrm{Q}}\left[\phi, A\right] \tag{1}$$

The first term describes the free evolution of the classical device, while the second one describes the Yang–Mills environment and gives account of the interaction between both gauge and classical fields. Fields A can be freely thought as standing for gauge fields and for any other quantum matter fields, but since the consideration of other quantum degrees of freedom is straightforward, we will explicitly exclude them.

Therefore, we assume that

$$\mathcal{S}_{\mathrm{Q}} = \mathcal{S}_{\mathrm{YM}}\left[A\right] + \lambda \mathcal{S}_{\mathrm{INT}}\left[\phi, A\right] \tag{2}$$

$\mathcal{S}_{\mathrm{YM}}$ being the pure Yang–Mills free action and $\mathcal{S}_{\mathrm{INT}}$ an interaction term. We have introduced the real number λ in order to parametrize the interaction strength, which we will assume to be weak.

Gauge–fixing is implemented by introducing into the classical action a gauge–breaking term depending on the gauge fields. According to the BRST–quantization scheme, in order to construct this term it is useful to extend the field–configuration manifold by considering three new fields: the Fadeev–Popov fermionic ghost and anti–ghost fields, ω and ω^*, and the bosonic auxiliary Nakanishi–Lautrup fields, h, all of them carrying the group index [14].

Although the gauge–fixing term is not invariant under gauge transformations, it is required to be a BRST–invariant functional on the extended field–space. BRST–symmetry is defined in terms of a non–linear operator s, the so called Slavnov operator, according to

$$\delta_{\mathrm{BRST}} = \epsilon s \tag{3}$$

where ϵ is an infinitesimal (global) fermionic parameter and the Slavnov operator acts on the fields in the following way:

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[1]Unless noted otherwise, we will use De Witt's condensed notation: indices will stand for all attributes of the fields and summation and integration over repeated indices will allways be understood. In addition, we will omit any kind of index if not strictly necessary. For example, we will write A instead of $A^a_\mu(x)$.

$$s\phi_\alpha = i\omega_a \left[t_a\right]_{\alpha\beta} \phi_\beta \tag{4}$$

$$sA_{a\mu} = \left(D_\mu \omega\right)_a = \partial_\mu \omega_a + f_{abc} A_{b\mu} \omega_c \tag{5}$$

$$s\omega_a = -\frac{1}{2} f_{abc} \omega_b \omega_c \tag{6}$$

$$s\omega_a^* = -h_a \tag{7}$$

$$sh_a = 0 \tag{8}$$

Here the t_a are matrices carrying the fully reducible representation of the gauge group when acting on matter fields, f_{abc} are the group structure constants and D_μ is, as usual, the covariant derivative in the adjoint representation.

If gauge–fixing at a classical level is imposed through conditions

$$f\left[A\right] = 0 \tag{9}$$

the gauge–fixing action reads

$$\mathcal{S}_{\rm GF}\left[A, \omega, \omega^*, h\right] = h_a f_a + \frac{1}{2}\alpha h_a h_a + \omega_a^* \frac{\delta f_a}{\delta A_{b\mu}} \left(D_\mu \omega\right)_b \tag{10}$$

where α is an arbitrary parameter [2]. The gauge–fixed action or classical effective action is, therefore,

$$\mathcal{I}\left[\phi, A, \omega, \omega^*, h\right] = \mathcal{S}_{\rm CD} + \mathcal{S}_{\rm YM} + \lambda \mathcal{S}_{\rm INT} + \mathcal{S}_{\rm GF} \tag{11}$$

BRST–transformations acting on the physical sector are simply reparametrizations of gauge transformations, implying that the gauge–invariant classical action $\mathcal{S}$ is also BRST–invariant. In fact, Fadeev–Popov ghost fields were introduced in order to realize the local gauge–invariance of the classical action as a global symmetry of the theory when defined on both the physical and unphysical sectors.

Anti–ghost and Nakanishi–Lautrup fields, on the other hand, were introduced in order to ensure the Slavnov operator nilpotency, i.e.,

$$s^2 = 0 \tag{12}$$

and to define the gauge–fixing condition as a Slavnov-variation. In fact, it is straightforward to show that

$$\mathcal{S}_{\rm GF} = s\Psi\left[A, \omega, \omega^*, h\right] \tag{13}$$

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[2]Notice that when integrated out in the path integral, Nakanishi-Lautrup auxiliary fields h reproduce the usual gauge-breaking term, quadratic in the functions fixing the gauge, f.

where we have defined

$$\Psi\left[A,\omega^{*},h\right]=\omega_{a}^{*}f_{a}+\frac{1}{2}\alpha\omega_{a}^{*}h_{a} \tag{14}$$

These properties imply the BRST–invariance of the gauge–breaking term and, therefore, of the complete classical effective action.

The classical nature of the fields describing the device allows us, when calculating the connected vacuum persistence amplitude, to sum over all connected Feynman graphs excluding those diagrams involving loops with internal legs representing matter propagators. Its path integral representaton is, then, the usual one with integration just over gauge and unphysical fields:

$$\mathcal{W}\left[\phi,J\right]=-i\ln\int\left[d\chi\right]\ \exp i\left\{\mathcal{I}+J_{n}\chi_{n}\right\} \tag{15}$$

where, we are denoting by χ_n all fields but the classical ones, index n running over all attributes of the fields, and J are external sources coupled to the quantized fields.

The corresponding expression for the expectation value of an arbitrary functional $\mathcal{F}$ defined on the extended field–space is

$$\langle\mathcal{F}\rangle=e^{-i\mathcal{W}}\int\left[d\chi\right]\ \exp i\left\{\mathcal{I}+J_{n}\chi_{n}\right\}\ \mathcal{F} \tag{16}$$

As can be seen from the equation above, the free action for the classical fields has no direct effect on expectation values, since classical configurations are not integrated and the purely classical contribution to $\mathcal{I}$, $\mathcal{S}_{\rm CD}$, is cancelled by normalization. Therefore, for simplicity, we will neglect such contribution.

The quantum effective action is defined as the sum of all 1–particle irreducible graphs without loops with matter legs and, formally, it is given by the Legendre functional transform of $\mathcal{W}$,

$$\Gamma\left[\phi,\overline{\chi}\right]=\mathcal{W}-J_{n}\overline{\chi}_{n} \tag{17}$$

From Eq. (17) it follows that $\overline{\chi}$ represents the expectation value of field χ, since

$$\overline{\chi}_{n}=(-)^{\chi}\,\frac{\delta\mathcal{W}}{\delta J_{n}} \tag{18}$$

where functional derivatives are acting on the right and $(-)^{\chi}=1$ if the field is bosonic and $(-)^{\chi}=-1$ otherwise.

A path integral self–contained representation of the quantum effective action can be obtained by using

$$\frac{\delta\Gamma}{\delta\overline{\chi}_{n}}=-J_{n} \tag{19}$$

Introducing these expressions into Eq. (15) it follows that

$$\Gamma=-i\ln\int\left[d\chi\right]\ \exp i\left\{\mathcal{I}-\frac{\delta\Gamma}{\delta\overline{\chi}_{n}}\left(\chi_{n}-\overline{\chi}_{n}\right)\right\} \tag{20}$$

Expectation values in terms of the quantum effective action are thus given by

$$\langle \mathcal{F} \rangle = e{-}i\Gamma \int [d\chi] \ \exp i \left\{ \mathcal{I} - \frac{\delta \Gamma}{\delta \overline{\chi}_n} \left(\chi_n - \overline{\chi}_n \right) \right\} \mathcal{F} \tag{21}$$

We would like to compute the variation in the quantum effective action due to a change in the gauge–fixing conditions $f \rightarrow f' = f + \Delta f$ and $\alpha \rightarrow \alpha' = \alpha + \Delta\alpha$, or, in other words, $\Psi \rightarrow \Psi' = \Psi + \Delta\Psi$.

Let us label the modified quantum effective action and expectation values with a prime. Then, we have

$$\Gamma' = -i \ln \int [d\chi] \ \exp i \left\{ \mathcal{I} + s\Delta\Psi - \frac{\delta \Gamma}{\delta \overline{\chi}_n} \left(\chi_n - \overline{\chi}'_n \right) \right\} \tag{22}$$

where we have set the external sources to be the same as those switched–on for the unchanged gauge condition, i.e.,

$$\frac{\delta \Gamma'}{\delta \overline{\chi}'_n} = -J_n \tag{23}$$

Substracting Eq. (20) from Eq. (22) it follows that

$$\Delta\Gamma = -i \ln \langle \exp i \left\{ s\Delta\Psi \right\} \rangle + \frac{\delta \Gamma}{\delta \overline{\chi}_n} \Delta\overline{\chi}_n \tag{24}$$

where $\Delta\overline{\chi} = \overline{\chi}' - \overline{\chi}$.

Up to first order in $\Delta\Psi$, Eq. (24) reduces to

$$\Delta\Gamma = \langle s\Delta\Psi \rangle + \frac{\delta \Gamma}{\delta \overline{\chi}_n} \Delta\overline{\chi}_n \tag{25}$$

where now $\Delta\overline{\chi}_n$ denotes the first order variation of the mean fields.

The so called Nielsen identity is simply the explicit form of this last equation. Performing a BRST–transformation in Eq. (21) with $\mathcal{F} = \Psi$ and making use of the fact that BRST–transformations preserve volume in the configuration space (even when restricted to the quantum sector) it follows that

$$\langle \Delta\Psi \rangle = e^{-i\Gamma} \int [d\chi] \ \exp i \left\{ \mathcal{I}\left[\phi, \chi + \epsilon s\chi\right] - \frac{\delta \Gamma}{\delta \overline{\chi}_n} \left(\chi_n + \epsilon s\chi_n - \overline{\chi}_n \right) \right\} \Delta\Psi\left[\chi + \epsilon s\chi\right] \tag{26}$$

Now, using the BRST–invariance of the classical effective action,

$$\mathcal{I}\left[\phi, \chi + \epsilon s\chi\right] = \mathcal{I}\left[\phi, \chi\right] - \epsilon \frac{\delta \mathcal{I}}{\delta \phi_\alpha} s\phi_\alpha \tag{27}$$

we get

$$\langle \Delta\Psi \rangle = \left\langle \exp i \left\{ -\epsilon \frac{\delta \Gamma}{\delta \overline{\chi}_n} s\chi_n - \epsilon \frac{\delta \mathcal{I}}{\delta \phi_\alpha} s\phi_\alpha \right\} \left(\Delta\Psi + \epsilon s\Delta\Psi \right) \right\rangle \tag{28}$$

Expanding the exponential in Eq. (28), it follows that

$$\Delta\Gamma = i\frac{\delta\Gamma}{\delta\overline{\chi}_n}\left\langle\left(s\chi_n\right)\Delta\Psi\right\rangle + \frac{\delta\Gamma}{\delta\overline{\chi}_n}\Delta\overline{\chi}_n + i\left\langle\frac{\delta\mathcal{I}}{\delta\phi_\alpha}\left(s\phi_\alpha\right)\Delta\Psi\right\rangle \tag{29}$$

This is the generalized Nielsen identity we have announced in the introduction. It controls the gauge dependence of the quantum effective action and it is the starting point for the demonstration of the gauge–fixing independence of any observable magnitude. In the absence of classical fields, the third term in the r.h.s of Eq. (29) vanishes, and the usual Nielsen identity is recovered [1,2]. In that particular case (no classical fields) one can infer from the above identity the gauge–fixing independence of spontaneous symmetry breaking, Higgs masses [2], nucleation rates [15], etc.

III. GAUGE–FIXING INDEPENDENCE OF TEST FIELDS

We will now show that, up to first order in the interaction–strenght parameter λ, the dynamics of the classical fields turns out to be gauge–fixing independent when the mean values of the quantum fields are on–shell. That is, the explicit gauge–fixing dependence of the mean values cancels out the explicit dependence of $\Delta\Gamma$.

Keeping only first order terms in λ is justified by the fact that we are assuming that the classical field have no influence on quantum mean values. Under these conditions, that we will refer as the *test field conditions*, the quantum effective action has two main contributions:

$$\Gamma = \Gamma^0\left[\overline{\chi}\right] + \lambda\Gamma^1\left[\phi,\overline{\chi}\right] + \mathcal{O}\left(\lambda^2\right) \tag{30}$$

where Γ^0 is the quantum effective action for the free Yang–Mills background and Γ^1 involves those terms describing the evolution of matter and its interaction with gauge fields. In addition, the on–shell configurations of the quantized fields are those obtained by solving the free Yang–Mills equations of motion,

$$\frac{\delta\Gamma^0}{\delta\overline{\chi}}\left[\overline{\chi}^0\right] = 0 \tag{31}$$

Eq. (30) and Eq. (31) imply that we can freely add to the free on–shell mean fields $\overline{\chi}^0$ an arbitrary term linear in λ, since any first order correction to these configurations has a second order effect in the final expression of the quantum effective action for matter, i.e.,

$$\Gamma_{\text{on–shell}} = \Gamma\left[\phi,\overline{\chi}^0\right] \approx \Gamma\left[\phi,\overline{\chi}^0 + \lambda\xi\right] \tag{32}$$

where we have denoted by $\xi = \xi\left[\phi,\overline{\chi}^0\right]$ this arbitrary additional term. Accordingly,

$$\Delta\Gamma_{\text{on–shell}} = \Delta\Gamma\left[\phi,\overline{\chi}^0\right] \approx \Delta\Gamma\left[\phi,\overline{\chi}^0 + \lambda\xi\right] \tag{33}$$

The free Yang–Mills quantum effective action on–shell is gauge–independent. Moreover, the variation of the mean fields due to changes in the gauge–fixing conditions, $\Delta\overline{\chi}$, vanishes on–shell for the free case. Therefore, from Eq. (29) we obtain

$$\Delta\Gamma\left[\phi, \overline{\chi}^0 + \lambda\xi\right] = i\lambda\left[\frac{\delta^2\Gamma^0}{\delta\overline{\chi}_m\delta\overline{\chi}_n}\xi_m + \frac{\delta\Gamma^1}{\delta\overline{\chi}_n}\right]\langle(s\chi_n)\,\Delta\Psi\rangle^0 + i\lambda\left\langle\frac{\delta\mathcal{S}}{\delta\phi_\alpha}(s\phi_\alpha)\,\Delta\Psi\right\rangle^0 \tag{34}$$

Here derivatives of the effective action are evaluated at χ^0 and the superscript on the brackets means that expectation values are calculated neglecting the interaction between matter and gauge fields.

The functions ξ still remain arbitrary. So, it is possible to choose them such that they cancel the r.h.s of the equation above. With this choice it immediately follows that

$$\Delta\Gamma_{\text{on–shell}} = 0 \tag{35}$$

showing that the dynamics of the test field is, up to first order in λ, gauge–fixing independent.

IV. CONCLUSIONS

Being gauge–fixing dependent, the mean fields in Yang-Mills theories have no physical meaning. Therefore, to analyze quantum effects it is not enough to compute the effective action and solve the quantum corrected equations for the mean fields. It is also necessary to extract physical information from them. In this paper we have shown that a way of doing this is to couple classical test fields to the Yang-Mills fields. The dynamics of the test fields is gauge–fixing independent, and therefore physically relevant. These results are in tune with previous ones obtained in the context of semiclassical gravity [12].

In Ref. [13], the authors analyzed the same problem addressed here. Using a Slavnov identity for the vacuum persistence amplitude $\mathcal{W}$, they showed that the dynamics of the classical fields is gauge–fixing independent in the low energy limit of power-counting renormalizable theories. In other words, they showed that within these reasonable assumptions the third term in the r.h.s of Eq. (29) vanishes. In this paper we presented a different calculation based on Nielsen identity for the effective action. In our approach we did not make any assumption about renormalizability. This may be very useful for the analysis of the gauge–fixing independence in semiclassical gravity, since the results are apparently dependent on the classical device (see Refs. [12] and [16]).

We have not taken into account the divergences present in any field theory. All our formal calculations should be understood with a regularization that does not break gauge symmetry. Moreover, when considering classical fields, it may be necessary to add appropiate counterterms to the action of the classical field. We will present some specific examples in a forthcoming publication.

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